Announcements

1) Using Mathematica

after entering input,

need to hit

"Shift + enter" to

record the commands.

Also, right-hand enter

key alone will work.

- create notebook file

- Define a function by

$$f[x\_] := \cdots$$

- Evaluating at specific numbers, no underscore, just $f[2]$, $f[Pi]$, etc.

2) Turfe lecture

tomorrow, 3-4,

CB 1030

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Look at minimizing

$\|Ax - b\|_2$.

If $b \in \operatorname{ran}(A)$, the minimum is zero.

If not, the minimum occurs when Ax is the orthogonal projection of b onto $\operatorname{ran}(A)$.

Recall: normal equations

$$A^*Ax = A^*b$$

Since A is rank n, A^*A is invertible, and we get

$$x = \underbrace{(A^*A)^{-1}A^*}_{= A^+,\ \text{the pseudoinverse}} b$$

Then

$$y = Ax = AA^{+} b$$

You can check that

$AA^{+} = P$, the orthogonal projection onto $\operatorname{ran}(A)$.

y minimizes the 2-norm,

x = vector that maps to y.

Problem:

$$f: \mathbb{C}^{m \times n} \times \mathbb{C}^m \to \mathbb{C}^n$$

$$f(A, b) = x \quad \text{where}$$

$$x = A^+ b$$

Note: we can take the norm $\|(A,b)\| = \|A\|_2 + \|b\|_2$ on $\mathbb{C}^{m \times n} \times \mathbb{C}^m$, $\| \ \|_2$ on $\mathbb{C}^n$

— or —

$$g : \mathbb{C}^{m \times n} \times \mathbb{C}^{m} \to \mathbb{C}^{m}$$

$g(A, b) = y$ where

$$y = AA^{+}b \ (= Ax)$$

Note: we again let

$$\|(A, b)\| = \|A\|_2 + \|b\|_2,$$

take $\|\cdot\|_2$ on $\mathbb{C}^m$.

We will reduce to only changing either the input b or the input A at any time. We won't change both.

This gives 2 new problems.

New f:

$$f: \mathbb{C}^{m \times n} \longrightarrow \mathbb{C}^{n} \quad (x)$$

or

$$f: \mathbb{C}^{m \times n} \longrightarrow \mathbb{C}^{m} \quad (y)$$

We fix b,

$f(A) = x$ or

$f(A) = y$, $\| \ \|_2$ on all spaces

New g:

$$g: \mathbb{C}^m \to \mathbb{C}^n \quad (x)$$

or

$$g: \mathbb{C}^m \to \mathbb{C}^m \quad (y)$$

We fix A,

$g(b) = x$ (1st case)

$g(b) = y$ (2nd case),

$\|\cdot\|_2$ on all spaces.

Q: Are any of these problems well-conditioned?

A: Yes...

Example 1:

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A : \mathbb{C} \to \mathbb{C}^2$$

$$\operatorname{rank}(A) = 1$$

$$\operatorname{ran}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

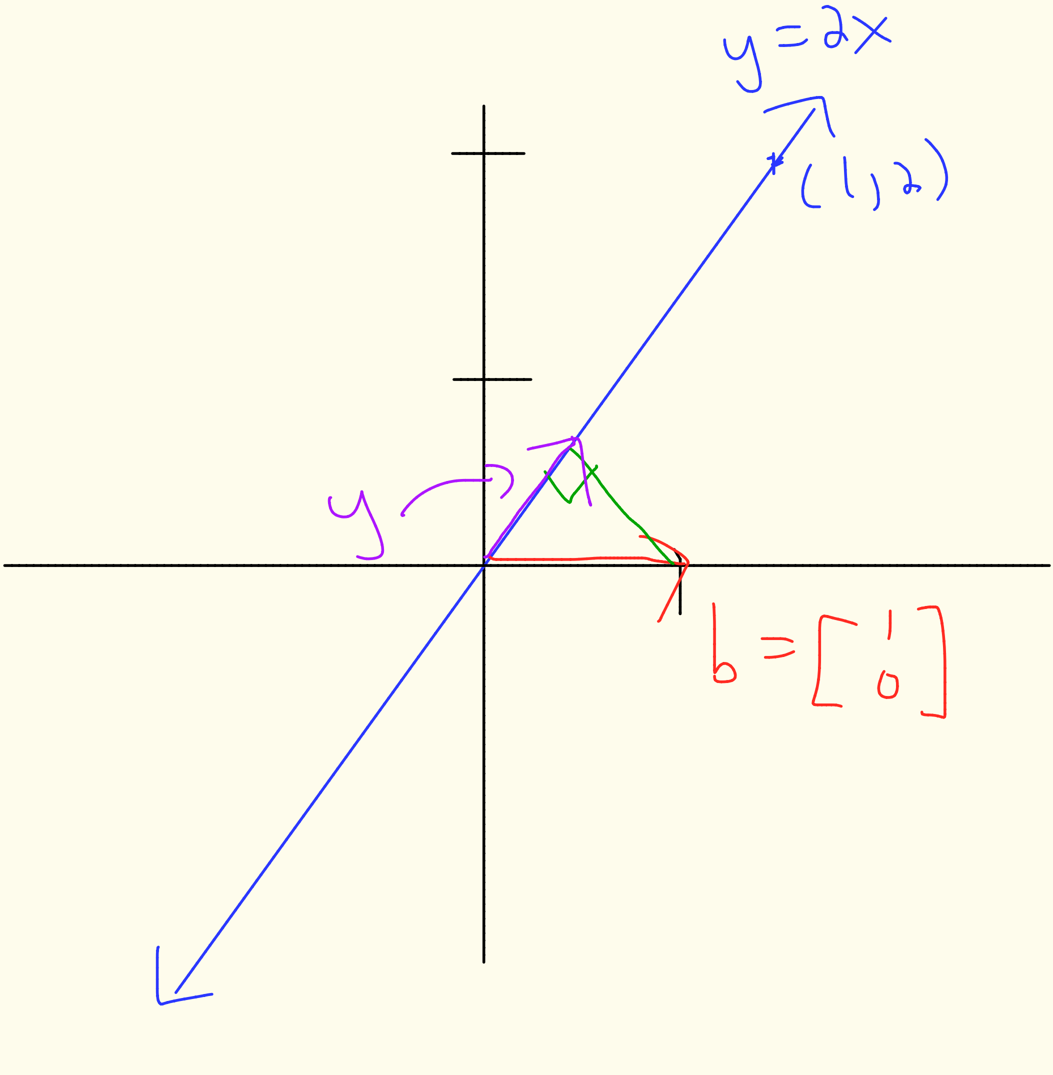
$$= \text{the line } y = 2x.$$

Use least-squares to "solve"

$$Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ not in } \operatorname{ran}(A) \right)$

Picture



Let $\tilde{A} = \begin{bmatrix} 1.0001 \\ 2 \end{bmatrix}$

$\tilde{b} = \begin{bmatrix} .9999 \\ 0 \end{bmatrix}$

We should expect the number $K(A) = \|A\|_2 \|A^+\|_2$ to show up since that's what appeared when solving $Ax = b$ with $A \in \mathbb{C}^{m \times n}$, invertible

Is it true that $K(A)$ is still the condition number for the least squares problem?

Numbers bigger than $K(A)$ are problematic for a positive answer.

First calculate $K(A)$.

We get $K(A) = 1$.

Let's first consider the problem of changing b to $\tilde{b}$ and then solving for x (and $\tilde{x}$).

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x = b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \tilde{x} = \tilde{b} = \begin{bmatrix} .9999 \\ 0 \end{bmatrix}.$$

Use least squares.

$$x = A^+ b, \quad x = 1/5$$

$$\tilde{x} = A^+ \tilde{b}, \quad \tilde{x} = .19998$$

Calculating:

$$\frac{|x-\tilde{x}|}{\|b-\tilde{b}\|_2} \cdot \frac{\|b\|_2}{|x|}$$

$$\|b\|_2 = 1$$

$$|x| = 1/5$$

$$|x-\tilde{x}| = .00002$$

$$\|b-\tilde{b}\|_2 = .0001$$

$$\frac{|x-\tilde{x}|}{\|b-\tilde{b}\|_2} \cdot \frac{\|b\|_2}{|x|}$$

$$= \frac{.00002}{.0001} \cdot \frac{1}{\left(1/5\right)}$$

$$= \frac{.0001}{.0001} = 1$$

$$= K(A)$$

Now try y:

$$y = Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tfrac{1}{5}$$

$$= \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$$

$$\tilde{y} = A\tilde{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot (.19998)$$

$$= \begin{bmatrix} .19998 \\ .39996 \end{bmatrix}$$

$$\|y\|_2 = \frac{1}{5}\left\|\begin{bmatrix}1\\2\end{bmatrix}\right\|_2$$

$$= \frac{1}{\sqrt{5}}$$

$$\|y-\tilde{y}\|_2 = \left\|\begin{bmatrix}.00002\\.00004\end{bmatrix}\right\|_2$$

$$= .00002\left\|\begin{bmatrix}1\\2\end{bmatrix}\right\|_2$$

$$= \sqrt{5}\,(.00002)$$

Calculating:

$$\frac{\|y-\tilde{y}\|_2}{\|b-\tilde{b}\|_2} \cdot \frac{\|b\|_2}{\|y\|_2}$$

$$= \frac{\sqrt{5}\,(.00002)}{.0001} \cdot \frac{.1}{\left(\frac{1}{\sqrt{5}}\right)}$$

$$= 1$$ ✓